An Experiment in the Aesthetic Value of Sonified Mathematical Objects

Dissertation Defense

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Review of the Golden Ratio and Fibonacci Sequence

We define the division of a line segment AC to be by GR when we divide it at the point B such that the resulting line segments have the property BC:AB = AB:AC.

GR is thus defined to be a pair of inversely related values: $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$. The first known mention of GR was by Euclid in his *Elements* (c. 300 BCE) [1].

We define a GFS G_n to begin with two integers $G_0 = a$ and $G_1 = b$. Values for $0 > n > 1$ are given by the Fibonacci rule $G_n = G_{n-1} + G_{n-2}$. The classic Fibonacci sequence begins $< 0, 1, 1, 2, 3, 5, 8, 13, \cdots >$. A method for its formation was given by Virahāṅka (c. 600–800 CE) [2].

Review of the Golden Ratio and Fibonacci Sequence

GFS is closely related to GR: the ratio of the adjacent values $G_n and G_{n+1}$ converges to GR as we approach the limit.

If we take the Fibonacci sequence, for example, and use adjacent values for the sides of a rectangle, we approach the "golden rectangle."

Overview

Adolf Zeising (1810–1876)

Adolf Zeising's name is closely associated with the concept of *golden numberism* [3].

Zeising wrote three important works that outlined his views on aesthetics and the role of GR in nature and art [4, 5, 6]:

- *Neue Lehre von den Proportionen des menschlichen Körpers* [New Doctrine on the Proportions of the Human Body] (1854)
- *Aesthetische Forschungen* [Aesthetic Research Studies] (1855)
- *Das Normalverhältniss der chemischen und morphologischen Proportionen* [The Standard Ratio of Chemical and Morphological Proportions] (1856).

Zeising's Claims

- Proportionality's connection to beauty.
- GR as a previously unknown fundamental morphological law pervading all of nature and art.
- GR underlying the proportionality of the human body, leaf arrangement on plants, proportions of architectural and sculptural works of art, aesthetically pleasing chords in musical harmony, etc.
- Nature approximates GR.
- GR is not the only ratio of aesthetic value.

Zeising rarely provided scientific evidence for his claims.

There was nonetheless great interest in Zeising's claims among scientists, philosophers and artists.

Gustav Theodor Fechner (1801–1887)

Fechner's work in experimental aesthetics began with experiments designed to determine the aesthetic significance of GR in simple geometric objects.

Fechner's contributions to aesthetic theory [7, 8, 9]:

- The *bottom-up approach* vs. *top-down*.
- *Direct impressions* vs. *associative impressions*.
- *Direct* vs. *associative factors* of the impression and *associative influence*.
- Aesthetic impressions' modification through constitution or combination with other equal or different forms: *combinatorial influence*.
- Necessity of simple examples.
- Polling as a method of measuring aesthetic value. Aesthetic value based on empirical observation.
- Three experimental methods: the *method of choice*, *method of creation*, and *method of utilization*.

Experiments in the Visual Aesthetic Value of GR

- Fechner confirms GR's aesthetic value as aspect ratio (1860s) [9].
- R. Angier questions experimental method (1903) [10].
- C. Lalo's experiments confirm Fechner's (1908) [11].
- E. Thorndike's experiments call Fechner and Lalo's results into question (1917) [12].
- P. Farnsworth's results support the conclusions of Fechner and Lalo (1932) [13].
- W. C. Shipley, P. E. Daltman, and B. A. Steele determine an average preference for 1.54:1 (1947) [14].
- C. W. Nienstedt and S. Ross determine a preference for GR among college students (1951) [15].
- M. Godkewitsch concludes that preference for GR is an artifact of its position in the range of stimuli (1974) [16].

Thesis Statement

Motivation: It is part of human nature to desire to understand what we find beautiful and why.

Considering that significant interest has been taken in determining the aesthetic value of GR in the visual domain, it is surprising that no parallel experimental research has ever been carried out in the audio domain.

Do sonifications of mathematical objects closely related to FS and GR hold some special aesthetic significance or advantage over less closely related sonified mathematical objects?

Experiment Description

Independent variable: the method of creating the stimuli

Dependent variable: aesthetic preference in two-alternative forced-choice (2AFC) tests

Sample: 165 subjects (18–67 years old, ♀ 63.6% and ♂ 36.4%)

Procedure:

- Five pairs of sonifications of mathematical objects were created. In each pair, one sonified mathematical object was closely related to GR and FS, whereas the other was not.
- The five pairs were each presented three times to subjects in pseudo-random order.
- Subjects were asked to choose which sonification in a pair they found more aesthetically pleasing.

Results: Fibonacci-related mathematical objects have a negative effect on aesthetic value in sonification.

Additional Research Questions

Which sonification is more aesthetically pleasing in each pair?

Given that each pair is presented three times, are subjects more consistent in their choices than would occur by chance?

GR and FS in Music

Many composers have used GR and FS in their music: Bartók, Xenakis, Nono, Krenek, Stockhausen, Nørgård, Chowning, Ferneyhough, Barlow, etc.

Applications are multifarious: to pitch, temporal proportions, meter, etc.

None of these composers' works are suitable for our research purposes: there are too many other influences in the compositions and nothing suitable to compare them to.

Direct sonification of mathematical objects is the best method of creating the necessary stimuli for research purposes.

Design Criteria

In order to limit combinatorial influence and create fair comparisons, the following criteria were met for the sonifications chosen for the study:

- Analog mathematical objects were chosen for each pair: a binary sequence was compared to another binary sequence, a signature sequence was compared to another signature sequence, etc.
- Fair comparisons were sought: I chose mathematical objects not related to GR and FS that I found to be interesting both mathematically and when sonified.
- Various types of mathematical objects were chosen: binary sequences, real number sequences, numeral systems, integer sequences, and single irrational numbers.
- The sonifications were carried out in an identical manner within each pair: different numbers were simply "plugged in."
- The total time attributed to the Fibonacci-related sonifications was within 1% of the total time attributed to the sonifications they were pitted against.

Golden String vs. Thue-Morse Sequence (Pair 1/5)

The **golden string** (related to Fibonacci) can be generated by means of a substitution map:

$$0 \rightarrow 1$$
$$1 \rightarrow 10.$$

We then take the limit of the sequence that is created by starting with 0:

$$0 \rightarrow 1 \rightarrow 10 \rightarrow 101 \rightarrow 10110 \rightarrow 10110101 \rightarrow 1011010110110 \rightarrow \ldots$$

We can generate the **Thue-Morse sequence** (not closely related to Fibonacci) by means of a substitution map in a similar fashion [17]:

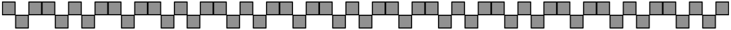
$$0 \rightarrow 01$$
$$1 \rightarrow 10.$$

We then take the limit of the sequence created starting with 0:

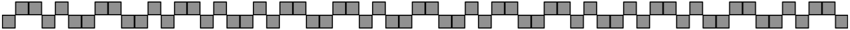
$$0 \rightarrow 01 \rightarrow 0110 \rightarrow 01101001 \rightarrow 0110100110010110 \rightarrow \ldots$$

Golden String vs. Thue-Morse Sequence (Pair 1/5)

golden string, first 55 values



Thue-Morse sequence, first 64 values



Golden String vs. Thue-Morse Sequence (Pair 1/5)

The sonification of these sequences was extremely simple:

- Each integer was attributed an equal duration of .109 seconds.
- 1s were placed on the right speaker, the 0s on the left.
- The same sample was used for the 1s and 0s, created from the sound of a single pop (a grain) from burning charcoal passed through a resonance filter (*reson* in Csound) with a center frequency of 256 and a bandwidth of 128.
- $F_{13} = 233$ members of the golden string were played; the duration of the sonification was 25.397 seconds. $2^8 = 256$ members of the Thue-Morse sequence were played; the duration was 27.904 seconds.

Golden String   Thue-Morse Sequence

Fractional Part of φ vs. $\frac{1}{8}$ Multiples (Pair 2/5)

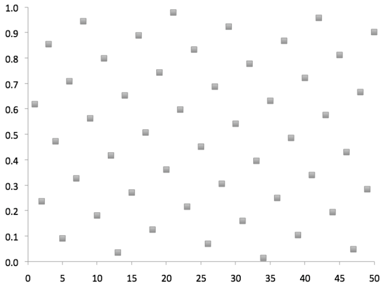
Here, **frac($n\varphi$)** is compared to **frac($n\frac{1}{8}$)**. The sequences begin as follows:

n	frac($n\varphi$)	frac($n\frac{1}{8}$)
1	.61803...	.125
2	.23607...	.25
3	.85410...	.375
4	.47214...	.5
5	.09017...	.625
6	.70820...	.75
7	.32624...	.875
8	.94427...	.0
9	.56231...	.125
↓	↓	↓

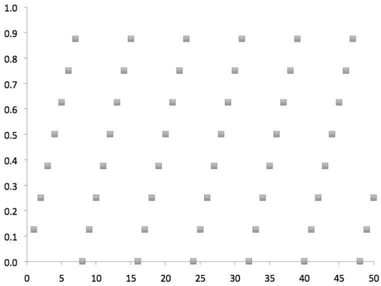
The sequence of frac($n\frac{1}{8}$) is periodic with period 8, whereas frac($n\varphi$) is aperiodic [18].

Fractional Part of φ vs. $\frac{1}{8}$ Multiples (Pair 2/5)

frac$(n\varphi)$



frac$(n\frac{1}{8})$



Fractional Part of φ vs. $\frac{1}{8}$ Multiples (Pair 2/5)

These sequences were also sonified exclusively in the spatial domain.

- Each number R was attributed an equal duration of .09375 seconds.
- A combination of amplitude panning and interaural time delay was used as a means of spatialization.
- A single impulse (of one sample) was used to represent each number.
- Numbers $R < .5$ emanated from the left side, $R = .5$ emanated exactly from the middle, and $R > .5$ emanated from the right side.
- Each sonification was played for exactly 20 seconds.

$n \times \varphi$ $n \times .125$

Dual Zeckendorf Representations vs. Binary Integers (Pair 3/5)

In the **binary numeral system**, $n \in \mathbb{N}$ is represented as

$$\sum_{j=0}^{M} c_n 2^n,$$

where $c_n \in \{0, 1\}$.

The **dual Zeckendorf representations** form a system of representing integers with FS as a base in which no consecutive 0s occur in a representation. In this system, $n \in \mathbb{N}$ can be represented as

$$\sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j + c_{j+1} \geqslant 1$ [19].

Dual Zeckendorf Representations vs. Binary Integers (Pair 3/5)

n	**dual Zeckendorf representation**	**binary numeral system**
1	1	1
2	10	10
3	11	11
4	101	100
5	110	101
6	111	110
7	1010	111
8	1011	1000
9	1101	1001
10	1110	1010
11	1111	1011
12	10101	1100
↓	↓	↓

Dual Zeckendorf Representations vs. Binary Integers (Pair 3/5)

dual Zeckendorf representations $1 \geqslant n \leqslant 143$

binary numeral system $1 \geqslant n \leqslant 128$

Dual Zeckendorf Representations vs. Binary Integers (Pair 3/5)

The dynamic parameters utilized were: simulated spatial location, frequency, and loudness.

- Each representation was attributed an equal duration of .27 seconds.
- Karplus-Strong string synthesis (*pluck* in Csound) was used.
- The series of harmonic partials with a fundamental frequency of 96 Hz was used. Bit numbers in each numeral system were mapped directly to the partials, beginning the bit number count at 1.
- Amplitude panning and delay was used as a means of spatialization.
- Lower bit numbers emanated from the right side, higher ones from the left.
- Lower bit numbers were quieter, higher ones were louder.
- $F_{12} - 1 = 143$ members of the dual Zeckendorf representations were played; the duration of the sonification was 38.61 seconds. $2^7 = 128$ members of the binary numeral system were played; the duration was 34.56 seconds.

Dual Zeckendorf Representations   Binary Integers

Signature Sequences: Φ vs. Ramanujan-Soldner Constant (Pair 4/5)

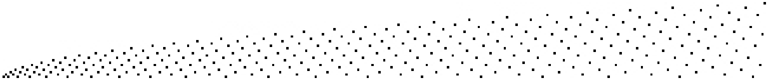
The Ramanujan-Soldner constant μ is defined as the positive root of the logarithmic integral $\mathrm{li}(x)$ for real x. It is an irrational number with an approximate value of $\mu \approx 1.4513692349$ [17].

Given positive integers i and j, if R is a positive irrational number and we arrange the set of all numbers $i + jR$ in order of increasing value, $i_1 + j_1 R$, $i_2 + j_2 R$, $i_3 + j_3 R$, $\ldots$, then $< i_1, i_2, i_3, \cdots >$ is the signature sequence of R [20].

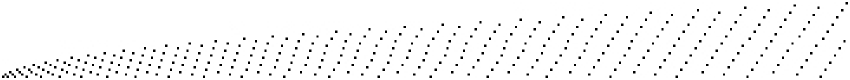
The **signature sequence of Φ** is characterized by great uniformity in spacing, whereas the **signature sequence of μ** is less uniform in this regard.

Signature Sequences: Φ vs. Ramanujan-Soldner Constant (Pair 4/5)

signature sequence of Φ (323 members)



signature sequence of μ (358 members)



Signature Sequences: Φ vs. Ramanujan-Soldner Constant (Pair 4/5)

The dynamic parameters utilized in these sonifications were: simulated spatial location, frequency, and loudness.

- Each representation was attributed an equal duration of .068 seconds.
- Wavetable synthesis was used; the wavetable used was created with a viola.
- The tuning utilized was the series of harmonic partials. A fundamental frequency of 94 Hz was chosen. The integers in the sequence were mapped directly to the partial numbers.
- Amplitude and delay were used as a means of spatialization.
- Lower integers emanated from the left side, higher ones from the right.
- Lower integers were quieter, higher ones were louder.
- 323 members of the signature sequence of Φ were played; the duration of the sonification was 21.964 seconds. 358 members of the signature sequence of μ were played; the duration was 24.344 seconds.

Φ   Ramanujan-Soldner Constant

Φ vs. $\sqrt{3}$ (Pair 5/5)

As continued fractions, $\mathbf{\Phi}$ and $\sqrt{\mathbf{3}}$ can be expressed as

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}} \quad \text{and} \quad 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}}},$$

respectively [21]. $\Phi \approx 1.6180339888$ and $\sqrt{3} \approx 1.7320508076$.

Φ vs. $\sqrt{3}$ (Pair 5/5)

Steady pulses were used with a tempo ratio between the speakers of 1:Φ for one sonification, and 1:$\sqrt{3}$ for the other.

- The left speaker in each sonification held a steady pulse of .32 seconds. The right speaker held a steady pulse of $.32 \times \Phi \approx .517770876$ seconds for one sonification, and $.32 \times \sqrt{3} \approx .554256258$ seconds for the other.
- Both speakers started simultaneously.
- The same sample of a single strike of a drum stick on a Remo drum pad was used on both speakers at the same amplitude.
- Each sonification was played for exactly 21 seconds.

Φ $\sqrt{3}$

Two Approaches to Data Analysis

There are two primary approaches to analyzing the data gathered:

1. We can look at the data irrespective of a subject's consistency.
2. We can discard any results where a subject's choice was not consistent.

We will utilize both of these methods and compare results.

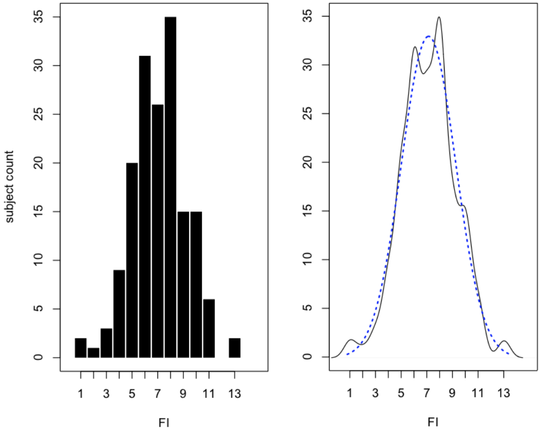
Aesthetic Value of FS and GR

The number of times a Fibonacci-related sonification is chosen by a subject is called the *Fibonacci index* (henceforth FI).

The maximum value for FI is $15 = 5 \times 3$; the minimum is 0.

Aesthetic Value of FS and GR

Distribution of FI for the Sample of 165 Subjects



mean value = 7.12

Single Sample Wilcoxon Signed-Rank Test

The single sample Wilcoxon signed-rank is designed to test a hypothesis about the *median* (or mean) of a population distribution.

We make the following assumptions in conducting a Wilcoxon signed-rank test:

- The sample has been randomly selected from the population it represents.
- The form of the values obtained in the test is that of interval/ratio data.
- The underlying population has a symmetric distribution. Under this assumption, the median equals the mean.

Definitions

Several important concepts in statistics:

- The *level of significance* is the probability of rejecting the null hypothesis when it is in fact true.
- We will use a level of significance of $\alpha = .05$.
- The *p-value* indicates the probability of obtaining a test statistic at least as extreme as the one that was actually observed strictly by chance.
- In order to reject a null hypothesis, we need a p-value less than α.

We will use these two numbers in determining whether to accept or reject our hypotheses.

Aesthetic Value of FS and GR

We will assume for our population of UCSB students, faculty and staff, that FI has a mean value $\mu = 15/2 = 7.5$:

Null hypothesis H_0: $\mu = 7.5$ (The use of Fibonacci-related mathematical objects in sonification has no impact on the aesthetic pleasantness of the resulting sonification.)

Alternative hypothesis H_a: $\mu \neq 7.5$ (The use of Fibonacci-related mathematical objects in sonification has either a positive or negative impact on the aesthetic pleasantness of the resulting sonification.)

As mentioned, we will use a level of significance of $\alpha = .05$.

Aesthetic Value of FS and GR

We observed a mean value of 7.12 for FI.

The p-value is $p = .021$ for the Wilcoxon signed-rank test.

$.021 < \alpha$, so we reject the null hypothesis, H_0: $\mu = 7.5$.

The use of Fibonacci-related mathematical objects in sonification appears to have a negative impact on the aesthetic pleasantness of the resulting sonification for our population.

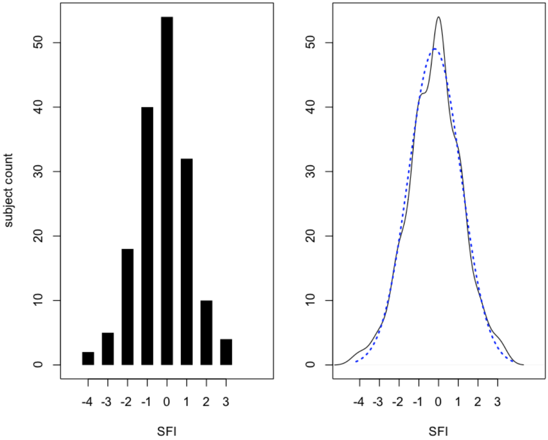
Aesthetic Value of FS and GR

The *secondary Fibonacci index* (henceforth SFI) only takes into account the pairs for which a subject was consistent.

SFI is calculated as the number of pairs for which a subject consistently chose the Fibonacci-related sonification minus the number of pairs for which a subject consistently chose the sonification not closely related to FS.

For example, if a subject chose one Fibonacci related pair and two pairs not closely related to FS, then SFI $= 1 - 2 = -1$. SFI ranges from -5 to 5.

Distribution of SFI for the Sample of 165 Subjects



mean value $= -.21$

Aesthetic Value of FS and GR

In this case, $p = .045$.

We reject the null hypothesis, H_0: $\mu = 0$, because $.045 < \alpha$.

We have obtained similar results as for FI.

The use of Fibonacci-related mathematical objects in sonification appears to have a negative impact on the aesthetic pleasantness of the resulting sonification for our population.

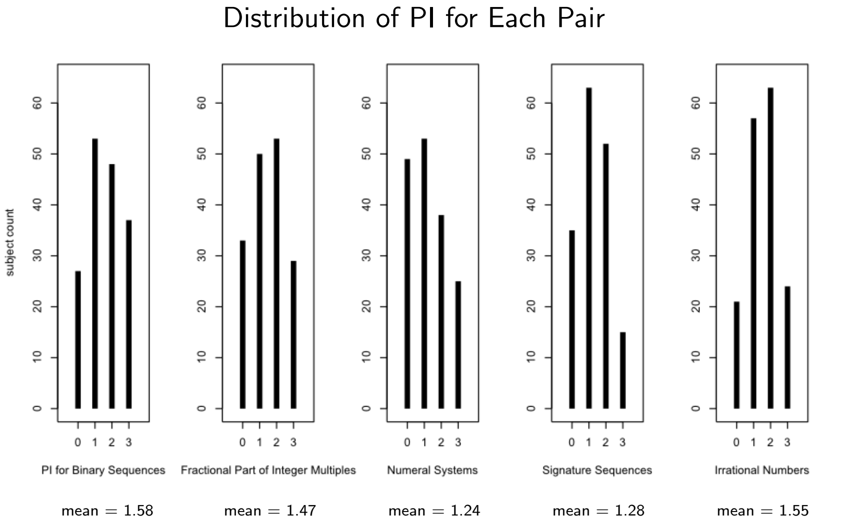
Individual Pairs

The number of times a Fibonacci-related sonification is chosen in a single pair by a subject is the *pair index* (henceforth PI).

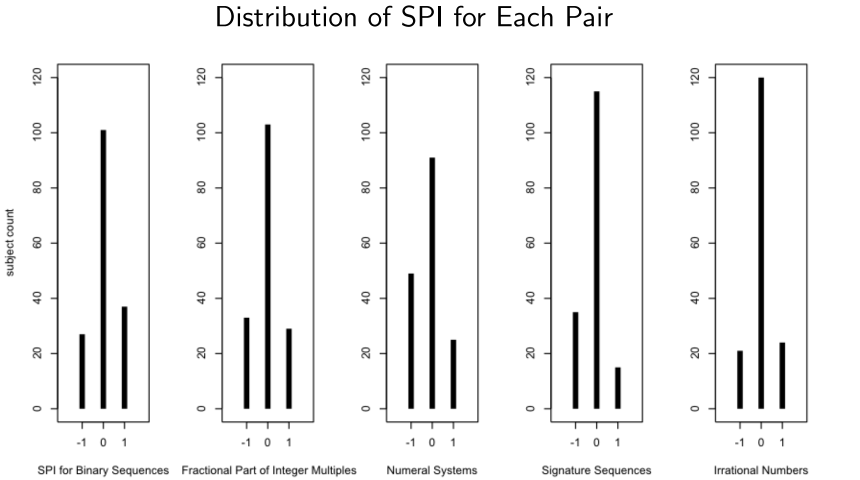
PI has a maximum value of 3 and a minimum of 0, as each pair is repeated three times.

The *secondary pair index* (henceforth SPI) is 0 if a subject was inconsistent on a pair, 1 if a subject chose the Fibonacci-related sonification consistently, and -1 if the subject chose the sonification not closely related to FS consistently.

Individual Pairs



Individual Pairs



mean = .06 | mean = −.02 | mean = −.15 | mean = −.12 | mean = .02

Individual Pairs

Binary Sequences

For PI, null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

For PI, $p = .366$.

We fail to reject the null hypothesis, H_0: $\mu = 1.5$, as $.366 > \alpha$.

For SPI, null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

For SPI, $p = .26$.

We achieve similar results as for PI and fail to reject the null hypothesis, because $.26 > \alpha$.

Fractional Part of Integer Multiples

For PI, $p = .753$.

We fail to reject the null hypothesis, H_0: $\mu = 1.5$, as $.753 > \alpha$.

For SPI, $p = .704$.

We achieve similar results as for PI and fail to reject the null hypothesis, because $.704 > \alpha$.

Individual Pairs

Numeral Systems

For PI, $p = .001$.

We reject the null hypothesis, as $.001 < \alpha$.

For SPI, $p = .007$.

We achieve similar results as for PI and reject the null hypothesis, as $.007 < \alpha$. **With the utilized method of sonification, it appears that the sonification of the binary numeral system is more aesthetically pleasing than the analog sonification of the (Fibonacci-related) dual Zeckendorf representations for our population.**

Dual Zeckendorf Representations Binary Integers

Individual Pairs

Signature Sequences

For PI, $p = .003$.

We reject the null hypothesis, as $.003 < \alpha$.

For SPI, $p = .007$.

We achieve similar results as for PI reject the null hypothesis: $.007 < \alpha$. **The sonification carried out for this study of the Ramanujan-Soldner constant appears to be more aesthetically pleasing than the analog sonification of Φ for our population.**

Φ Ramanujan-Soldner Constant

Individual Pairs

Irrational Numbers

For PI, $p = .505$.

We fail to reject the null hypothesis, as $.505 > \alpha$.

For SPI, $p = .766$.

We achieve similar results as for PI and again fail to reject the null hypothesis, as $.766 > \alpha$.

Subjects' Consistency

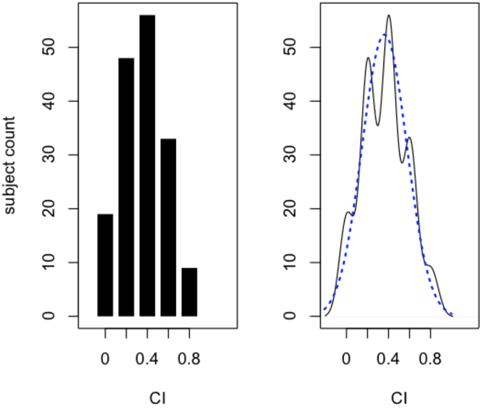
Each pair was presented three times.

The *consistency index* (henceforth CI) is given in percent and represents for how many pairs a subject chose the same sonification every time.

There are five pairs of sonifications, so CI is always either 0%, 20%, 40%, 60%, 80%, or 100% for an individual subject.

Subjects' Consistency

Distribution of CI for the Sample



mean = 35.8%

For any individual pair, the probability of being consistent in choice purely by chance is 25%, as there are $2^3 = 8$ distinct possibilities of choices (**000**, 001, 010, 011, 100, 101, 110, **111**).

Null hypothesis H_0: $\mu = .25$

Alternative hypothesis H_a: $\mu \neq .25$

For this test, $p \approx 0$ (2.98e−9).

Thus we reject H_0. **Subjects appear to be significantly more consistent in their choices of which sonification in a pair is more aesthetically pleasing than would occur purely by chance.**

Summary of Results

13 Wilcoxon signed-rank tests were carried out; seven yielded statistically significant results.

In all six cases having to do with the aesthetic value of GR and FS, all six leaned away from GR and FS (the seventh case involved subjects' consistency).

Based on our study sample, we can summarize the results obtained for our population of UCSB students, faculty and staff, as follows:

- Use of Fibonacci-related mathematical objects in sonification appears to have a negative impact.
- The sonification of the binary numeral system is more aesthetically pleasing than that of the dual Zeckendorf representations.
- The signature sequence of the Ramanujan-Soldner constant is more aesthetically pleasing than the analog sonification of Φ.

Interpretation of the Results

Many who are familiar with my artistic work assumed that I was going into this experiment in an attempt to provide evidence for the aesthetic value of GR and FS in music.

We might speculate the following concerning the results:

- Despite best efforts to choose five contrasting sonifications sonified within each pair in an identical manner, the results could be an artifact of the particular pairs sonifications chosen. Another possible explanation: the uniformity created by GR comes across in general as boring in the audio domain.
- As to why the sonification of the binary numeral system was found to be more aesthetically pleasant: perhaps because of the regularity and more clearly discernable symmetry of powers of 2.
- As for the signature sequences: perhaps because its pattern was less uniform than that of Φ.
- We failed to reject the null hypothesis for three of the individual pairs: either sample not large enough or there is no discernable difference between them.
- CI of 35.8% can be interpreted as a reflection of the discernability of the sonifications in individual pairs.

Conclusion

Humans desire to find out what they find to be aesthetically pleasing.

Researchers have been investigating FS and GR in the visual domain for many decades.

There was previously no parallel research in the audio domain.

The work carried out here provided initial insight into the question of the aesthetic value of GR and FS in the audio domain.

Limitations of this research:

- The question of the aesthetic value of GR and FS in the audio domain is dauntingly general: infinite number of sonifications, contradictory results in the visual domain.
- The same degree of accuracy in results cannot be obtained in the field of aesthetics as compared to various other scientific disciplines.

Conclusion

The research carried out in this dissertation could provide the starting point for further studies in this area:

- Would we obtain similar results about the potentially negative impact of the use of Fibonacci-related mathematics with different sonifications of the same mathematical objects?
- Would we obtain similar results with different mathematical objects?
- What would the results be if one utilized the method of creation in an experiment in the audio domain?
- What is the effect of knowledge about the mathematical objects sonified on the perceived aesthetic value of the sonification?
- Would we see differences in the results if we provided synchronized visualizations with the sonifications?

Conclusion

Thank you.

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